Reformulation of QCD in the language of general relativity

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Abstract

It is shown that there exists such collection of variables that the standard QCD Lagrangian can be represented as the sum of usual Palatini Lagrangian for Einstein general relativity and the Lagrangian of matter and some other fields where the tetrad fields and the metric are constructed from initial $SU(3)$ Yang - Mills fields.

1 Introduction

Unified description of all interactions is one of the main goals of the modern physics. Partial unification, namely unification of electromagnetic and weak interactions , is achieved in Salam - Weinberg theory and its numerous modifications. More or less satisfactory unification of electromagnetic, weak and strong interactions is achieved in grand unified theories based on various "large" gauge groups ($SU(5)$, $SO(10)$, etc.) But the satisfactory unified description of electromagnetic, weak, strong, and gravity interactions is still open problem.

The origin of the difficulties is clear. Whereas all realistic theories of strong, weak, and electromagnetic interactions are based on Yang - Mills (YM) action

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$$S_{YM} = \int \mathrm{tr}(dA + A \wedge A) \wedge *(dA + A \wedge A), \tag{1}$$

the general relativity is based on Einstein - Hilbert action

$$S_{EH} = \int dx \sqrt{g} R \tag{2}$$

or, in Palatini formalism, on the action [1]

$$S_P = \int e^a \wedge e^b \wedge (d\,\Gamma + \Gamma \wedge \Gamma)^{cd} \varepsilon_{abcd} \tag{3}$$

Obviously, that the mathematical structure of the action (1) and the actions (2) or (3) is very different. So the origination of the theory, that reduces to (1) and (2) (or (3)) in certain limiting cases is a very hard problem.

The most direct way to construct unified theory of all interactions is, of course, to replace the action (1) by some gravity-like action, or, vice-a-versa, to replace the action (2) or (3) by another one, that is more similar to (1).

The first possibility is realized, for instance, in tensor dominance (or strong gravity) model [6] (see also [2] for review and further references.) The Lagrangian of this model is very similar to gravitational one, but till now relation of this model and realistic physical models based on YM action is unclear.

The second possibility is realized, for example, in Poincare gauge theories of gravitation (see [1, 2] for review), or in $SL(6, C)$ gauge theory of Salam, Isham, and Strathdee [3] and their modifications [2, 4, 5]. But physical meaning of all above mentioned theories is not quite clear because the corresponding actions are unlike the action of Standard Model and it is not obvious that the latter can be considered as some limiting case of the former.

There exist also many other approaches to unification of gravity and YM gauge theories based on different modifications of actions (1)-(3). But, to author's knowledge, all theories proposed are rather far from real physics.

But there exist the third way to unification of general relativity and YM theories. Namely, one can try to find such variables that standard Einstein - Hilbert or Palatini actions written in these variables are transformed in standard YM action (plus, may be, the action with some supplementary

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[1]We omit insufficient overall factors before actions (1)-(3)

fields), or, vice-a-versa, one can try to transform by change of variables the usual YM action in Einstein - Hilbert or Palatini ones.

During the last twenty years, and especially during last five years, the great progress was achieved in both directions.

First of all, author would like to mention the Ne'eman - Sijacki "chromogravity" approach to QCD developed in papers [21]. Ne'eman and Sijacki showed that there exists the mechanism of appearance of gravity-like forces in infrared limit of QCD. Some speculations in spirit of Ne'eman - Sijacki approach were also given in recent paper of Kuchiev [22].

But in present paper we will follow another approach, namely the approach, proposed in the author's paper [7].

Let us consider, first, YM theory and general relativity in three dimensional space-time[2]. In this case YM action in the first order formalism can be written in the form

$$S_{YM} = \int \mathrm{tr} * F \wedge (dA + A \wedge A) + \lambda^2 \int \mathrm{tr} * F \wedge F, \qquad (4)$$

whereas Palatini action for gravity is

$$S_{3D} = \int e^a \wedge (d\Gamma + \Gamma \wedge \Gamma)^{bc} \varepsilon_{abc} \qquad (5)$$

In the formula (4) λ means coupling constant [3], and $*$ is the Hodge operator with respect to the space - time metric g_{mn}. For simplicity, below in this section we will consider the case of Euclidean space - time.

For $SU(2)$ gauge group, forms F and A valued in the space of anti-symmetric 3×3 matrices and so we can write

$$* F^{ab} = -\varepsilon^{abc} * F^c, \qquad (6)$$

$$S^{YM} = \int * F^a \wedge (dA + A \wedge A)^{bc} \varepsilon_{abc} - \lambda^2 \int * F^a \wedge F^a. \qquad (7)$$

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[2]Interesting approach to unification of YM theory and general relativity in three dimensions was proposed by Peldan [23]. However, it isn't clear, how to generalize this approach on 4D case.

[3]We reserve more usual notations e or g for determinants of the tetrad and the metric respectively.

In three dimensions $*F^a$ are 1-forms and so the first term in (7) coincides with three dimensional Palatini actions (5) up to notations! This fact allows to formulate 3D YM theory in general relativity-like form with the tensor

$$G_{mn} = (*F^a)_m (*F^a)_n \qquad (8)$$

as the new space - time metric. In particular, usual YM equations appear to be equivalent to Einstein ones with simple rhs.

Above mentioned results concerning relations between 3D gravity and 3D YM theory were obtained, first, in the author's paper [4] [7]. Independently, analogous results were obtained also in the work [8] in the context of (3+1) dimensional $SU(2)$ YM theory in the gauge $A_0^a = 0$. However, in the latter approach YM induced gravity lives only on the three dimensional hyperplanes $x^0 = const$ and so this approach is essentially non-covariant.

Further three dimensional space-time geometry discovered in works [7, 8] was investigated in papers [9, 10, 11, 12, 13, 14, 15]. In particular, in paper [9] solutions of Euclidean 3D YM equations with singularity on the sphere were discovered. These solutions can be also interpreted as stationary solutions of 4D YM equations in the gauge $A_0 = 0$ and can be considered as analog of Schwartzchild solution in general relativity. Analogous Schwartzchild-like and Kerr-like solutions of Yang - Mills - Higgs equations were recently discovered by Singlton [16].

It was shown that quantum particle moving in such YM field (that is considered as external one) inside this sphere can not leave it. So, may be, such solutions can be used for elaborating of black hole or microuniverse (see [17]) mechanism of confinement.

We see that in three dimensional world gravity does live inside YM theory. It is easy to understand the origin of such YM induced gravity. Indeed, the usual gravity is described by the triad of covectors e^a defined in each point of the space-time up to $SO(3)$ rotation, and $SO(3)$ connection Γ that defines the parallel transport of tensors in the space-time. All these objects appear naturally in YM theory – 1-forms $*F^a$ play the role of the triad and $SU(2)$ YM connection A plays the role of the space-time connection Γ.

[4]In some aspects, similar results were obtained by Halpern [30] in his investigations of self-dual sector of 4D YM theory. But Halpern's "metric" constracted from YM fields is not rank two space-time tensor and so it can not be considered as analog of the metric in general relativity.

But how to generalize this construction for realistic four dimensional case? Direct generalization is not possible, because, first, $*F$ in four dimensions are 2-forms (rather then 1-forms as in 3D case), and, the second, the structure of 4D Palatini action (3) differs from one of 3D action (5). Nevertheless, such generalization exists. Moreover, this problem was partially solved, in fact, almost twenty years ago in Plebanski's work [18]. But Plebanski obtained his results in absolutely different context (he investigated complex structures in general relativity). May be, due to this reason his results haven't been used in investigations of YM induced gravity till now[5].

Let us rewrite the Palatini action (3) in spinor notations [6] :

$$S_P = \int e^A_{\ C'} \wedge e^{BC'} \wedge (d\,\Gamma_{AB} + \Gamma_{AC} \wedge \Gamma^C_{\ B}) \tag{9}$$

One can note that 1-forms $e^{AA'}$ enter in action only in the combination

$$\Sigma^{AB} = e^A_{\ C'} \wedge e^{BC'} \tag{10}$$

So the Palatini action (9) can be represented in the form

$$S_P = \int \Sigma^{AB} \wedge (d\,\Gamma_{AB} + \Gamma_{AC} \wedge \Gamma^C_{\ B}) \tag{11}$$

Of course, the quantities Σ^{AB} in (11) cannot be considered as the independent dynamical variables. Indeed, due to (10) the 2-forms Σ^{AB} satisfy the condition

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0 \tag{12}$$

Further, Plebanski showed that if the conditions (12) are satisfied and

$$\Sigma^{AB} \wedge \Sigma_{AB} \neq 0, \tag{13}$$

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[5] After the completion of this paper author learned about very recent works by Robinson [33] in which the analogy between general relativity and YM theory in Plebanski approach is considerably clarified. The relations of results of [33] and ones obtained in the present paper need further investigations.

[6]We use the usual isomorphism between the spaces of $O(4)$ vectors and $SU(2) \times SU(2)$ spinors. Sign conventions, normalization factors, etc. are describe in the section 2 below. Further, we omit the part of Palatini action that contains the fields $\Gamma_{A'B'}$ because the full action is equivalent to the chiral action (9) . (See, for instance, Refs. [19, 20], in which it was shown that the using of chiral action (9) is very natural, in particular, in Ashtecar formalism.)

then Σ^{AB} can be represented in the form (10) with non-degenerate tetrad $e^{AA'}$.

So the Palatini action (9) is equivalent to Plebanski action

$$S_{Pl} = \int \Sigma^{AB} \wedge (d\,\Gamma_{AB} + \Gamma_{AC} \wedge \Gamma^C{}_B) + \int \phi_{ABCD}\Sigma^{AB} \wedge \Sigma^{CD} \qquad (14)$$

The second term in (14) with totally symmetric Lagrange multipliers ϕ_{ABCD} is introudced to take into account the condition (12).

In the action (14) fields Σ^{AB}, Γ_{AB} and ϕ_{ABCD} are independent dynamical variables. The first term in (14) coincides with the first term in the first order $SU(2)$ YM action

$$S_{YM} = \int F^{BC} \wedge (dA_{BC} + A_{BD} \wedge A^D{}_C) + \lambda^2 \int F^{BC} \wedge *F_{BC} \qquad (15)$$

up to notations. But it doesn't mean that the gravity lives inside $SU(2)$ YM theory as in 3D case, because the analog of the second term in (14) is absent in (15) and so we have no analog of (12) in $SU(2)$ YM theory. But without the condition (12) we cannot reconstruct the tetrad $e^{AA'}$.

Let us consider, however, the theory with more large gauge group $G \supset SU(2)$. One can choose among $N = \dim G$ 2-forms F three forms $F^{AB} = F^{(AB)}$ that are transformed as rank two symmetric spinor under gauge transformations from certain $SU(2)$ subgroup of the group G. Then the action (15) will be a piece of the total YM action. Further, if $\dim G \geq 8$, then, in general, we can impose, using other gauge degrees of freedom, five $SU(2)$ invariant gauge conditions

$$F^{(AB} \wedge F^{CD)} = 0 \qquad (16)$$

Conditions (16), that we will call "the Plebanski gauge", coincide with Plebanski conditions (12) up to notations whereas the first term in YM action (15) coincides, up to notations, with the first term in Plebanski action (14). *So we can conclude that gravity lives inside YM theory if the dimension of the gauge group is more or equal to eight.* Indeed, due to Plebanski theorem we can reconstruct the tetrad $e^{AA'}$ and the corresponding metric:

$$F^{AB} = e^A{}_{A'} \wedge e^{BA'} \qquad (17)$$

$$G_{mn} = e^{AA'}{}_m e_{AA'n} \qquad (18)$$

After substituting (17) in the first term of the action (15) we obtain the usual Palatini action for gravity (9).

The main idea of the present work is to use of the gauge (16) to reformulate the YM theory in general relativity-like form. Below we will show that the gauge (16) really exists for the gauge group $SU(3)$ and the corresponding gauge theory, the Quantum Chromodynamics, can be formulated in the close analogy with general relativity. But before author would like to give some additional notes concerning 2-forms formalism in general relativity.

Plebanski results allow to use three 2-forms Σ^{AB} instead of metric. In Plebanski's approach these forms satisfy the constraints (12) that play the crucial role in Plenanski's formalism. But later it was shown that this conditions aren't necessary. Namely, it appears that, in generic case, any three 2-forms define unique, up to conformal factor, the metric, with respect to which they are (anti)-self-dual. These statement is known now as Urbantke theorem (see [24]. Another proof and some refinements were given in [25]). In particular, in generic case any collection of three 2-forms defined up to $SL(3)$ transformation naturally determine the unique metric. Moreover, later 't Hooft showed [27] that any triple of two forms (with some non-degeneracy condition) naturally defines not only the metric but also certain $SL(3)$ connection and so it is possible to reformulate the general relativity in terms of triples of 2-forms. Similar formalism with $GL(3)$ connection instead of $SL(3)$ ones was proposed also in recent paper [26].

However, the Lagrangian of 't Hooft and its modifications are reduced to Plebanski's Lagrangian (14) by imposing of gauge conditions that are exactly coincide with (12). On the other hand, t' Hooft Lagrangian, without the imposing of conditions (12), is quadrilinear and so is not similar to YM one. By these reasons in the present paper we use the old Plebanski formalism rather then its further generalizations.

Clear relations between gravity and YM theory also appear in Ashtecar formalism [7]. Originally discovered [28], it was very unlike Plebanski approach. But later it was shown [29] that Ashtecar formalism can be reproduced by (3+1) decomposition of Plebanski Lagrangian.

The paper is organized as follows. In section 2 we describe our nota-

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[7]An attempt to develop the formalism for unified description of YM and gravity fields in the spirit of Ashtecar phase space approach was done in works [32]. However, this theory gives the conventional YM theory only in the lowest order in the fields and so it is hard to make consistent its predictions with ones of the Standard Model.

tions. In sections 3 and 4 we formulate QCD in general relativity-like form at classical amd quantum levels respectively. In section 5 we discuss obtained results.

2 Notations.

Indexes a, b, c, d are frame ones and run over the set $\{0, 1, 2, 3\}$. Indexes m, n, p, q are world ones and run over the same set. Upper case Latin indexes $A, B, C, ...$ are $SU(2)$ spinor ones and run over the set $\{0, 1\}$. Greek indexes α, β, γ runs over the set $\{1, 2, 3\}$

2.1 $SU(2)$ spinors and $O(3)$ vectors.

Lowering and raising of $SU(2)$ spinor indexes are performed by anti-symmetric spinors ε_{AB}, ε^{AB}, $\varepsilon_{01} = \varepsilon^{01} = +1$,

$$\varphi_A = \varphi^B \varepsilon_{BA}, \quad \varphi^B = \varepsilon^{BA} \varphi_A \tag{19}$$

Hermitian conjugation of $SU(2)$ spinors are defines as

$$(\varphi^\dagger)^{AB\cdots} = \bar{\varphi}^{CD\cdots} \varepsilon^{CA} \varepsilon^{DB} ... \tag{20}$$

Here the bar denotes complex conjugation. The spaces of symmetric second rank $SU(2)$ spinors and $O(3)$ vectors are isomorphic. The isomorphism is established by the formula

$$S^\alpha \longleftrightarrow S^{AB} = -\frac{i}{\sqrt{2}} S^\alpha \sigma_\alpha{}^{AB} \tag{21}$$

where $\sigma_\alpha{}^A{}_B$ are Pauli matrices. Real vectors correspond to Hermitian spinors,

$$\varepsilon^{\alpha\beta\gamma} U^\alpha V^\beta W^\gamma = \sqrt{2} U^{AB} V_{BC} W_A{}^C$$

$$S^\alpha S^\alpha = S^{AB} S_{AB}. \tag{22}$$

Below we will use the convention (21) with one exception: if Γ^α are components of some $O(3)$ connection and

$$R^\alpha = d\,\Gamma^\alpha + \frac{1}{2} \varepsilon^{\alpha\beta\gamma} \Gamma^\beta \wedge \Gamma^\gamma \tag{23}$$

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are components of the corresponding curvature form, then

$$\Gamma^{AB} = \frac{1}{2i}\sigma_\alpha{}^{AB}\Gamma^\alpha$$

$$R^{AB} = \frac{1}{2i}\sigma_\alpha{}^{AB}R^\alpha \tag{24}$$

Using (24), one can prove that

$$R^{AB} = d\,\Gamma^{AB} + \Gamma^A{}_C \wedge \Gamma^{CB} \tag{25}$$

2.2 $SU(2) \times SU(2)$ spinors and $O(4)$ vectors.

$O(4)$ frame vector indexes are lowered and raised by the tensor δ^{ab}. The spaces of rank (1,1) $SU(2) \times SU(2)$ spinors and $O(4)$ vectors are isomorphic. The isomorphism is established by the formula

$$S_{AA'} \longleftrightarrow S_a = g_a{}^{AA'}S_{AA'}, \tag{26}$$

where $g_a{}^{AA'}$ are Euclidean Infeld - van der Vaerden symbols for flat space:

$$(g_a{}^{AA'}) = \left(\frac{1}{\sqrt{2}}\delta^A{}_{A'}, \ \frac{i}{\sqrt{2}}\sigma_\alpha{}^A{}_{A'} \right) \tag{27}$$

Real vectors correspond to Hermitian spinors (Hermitian conjugation of the latter is defined in the previous subsection),

$$S_a S_a = S_{AA'}S^{AA'} \tag{28}$$

2.3 $O(4)$ (anti)-self-dual tensors and $O(3)$ vectors

For frame $O(4)$ tensors Hodge operator is defined as usual:

$$* M_{ab} = \frac{1}{2}\varepsilon_{abcd}M_{cd} \tag{29}$$

The spaces of the (anti)-self-dual tensors and $O(3)$ vectors are isomorphic. The isomorphism is established by the formula

$$^{\pm}M^\alpha = {}^{\pm}\eta^\alpha{}_{ab}{}^{\pm}M_{ab} \tag{30}$$

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where $^{\pm}\eta^{\alpha}{}_{ab}$ are 't Hooft symbols:

$$^{+}\eta^{\alpha}{}_{ab} = -\frac{1}{2i}\bar{\sigma}_{\alpha}{}^{A'}{}_{B'}g_{aAA'}g_b{}^{AB'} \tag{31}$$

$$^{-}\eta^{\alpha}{}_{ab} = \frac{1}{2i}\sigma_{\alpha}{}^{A}{}_{B}g_{aAA'}g_b{}^{BA'} \tag{32}$$

't Hooft symbols satisfy the following equations:

$$^{\pm}\eta^{\alpha}{}_{ac}{}^{\pm}\eta^{\beta}{}_{cb} = -\frac{1}{4}\delta^{\alpha\beta}\delta_{ab} + \frac{1}{2}\varepsilon^{\alpha\beta\gamma\pm}\eta^{\gamma}{}_{ab} \tag{33}$$

$$^{\pm}\eta^{\gamma}{}_{ab}{}^{\pm}\eta^{\gamma}{}_{cd} = \frac{1}{4}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \pm \varepsilon_{abcd}) \tag{34}$$

Formulas (22), (28), (33), and (34) allow to translate easily any formula from spinor to vector language and vice-a-versa.

3 Plebanski gauge in $SU(3)$ Yang-Mills theory

3.1 Plebanski theorem for real 2-forms

Plebanski showed that three complex 2-forms Σ^{AB}, satisfying the conditions (12) and (13), can be represented in the form (10). The "real" variant of this theorem can be formulated in the following way:

Let S^{α} be three real 2-forms obeying the conditions

$$S^{\alpha} \wedge S^{\beta} = \frac{1}{3}\delta^{\alpha\beta}S^{\gamma} \wedge S^{\gamma} \tag{35}$$

$$S^{\gamma} \wedge S^{\gamma} \neq 0 \tag{36}$$

Let G_{mn} be the Urbantke metric [8]

$$G_{mn} = -\frac{4}{3}\left[S^{\delta}{}_{tu}S^{\delta}{}_{vw}\varepsilon^{tuvw}\right]^{-1}\varepsilon_{\alpha\beta\gamma}\varepsilon^{pqrs}S^{\alpha}{}_{mp}S^{\beta}{}_{qr}S^{\gamma}{}_{sn} \tag{37}$$

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[8]Our definition of the metric (37) differs from the original definition of Urbantke [24] by insufficient factor.

Then G_{mn} has definite signature, $(++++)$ or $(----)$, and S^α can be represented, respectively, as

$$S^\alpha = \pm^- \eta^\alpha_{ab} e^a \wedge e^b. \tag{38}$$

One notes, that the equations (35) and (36) are nothing but reformulation of eqs. (12) and (13) in vector language. The spinor analog of (38) is

$$S^{AB} = \pm \frac{1}{2} e^{AC'} \wedge e^B_{C'} \tag{39}$$

where

$$(S^\dagger)^{AB} = S^{AB}, \quad (e^\dagger)^{AA'} = e^{AA'} \tag{40}$$

The eq. (39) is the analog of (10).

Let us prove the theorem formulated above. Let

$$M^{\alpha\beta} = \varepsilon^{mnpq} S^\alpha_{mn} S^\beta_{pq} \tag{41}$$

Then the matrix $M^{\alpha\beta}$ has a definite signature (see (35)). So, due to results of Urbantke [24] and Harnett [25], the Urbantke metric (37) is non-degenerate, has a definite signature, and 2-forms S^α are self-dual or anti-self-dual with respect to Hodge operator corresponding to this metric. Hence, the Urbantke metric can be written as

$$G_{mn} = \pm e^a_m e^a_n, \tag{42}$$

whereas 2-forms S^α as

$$S^\alpha = C^\alpha_\beta{}^- \eta^\beta_{ab} e^a \wedge e^b. \tag{43}$$

or

$$S^\alpha = C^\alpha_\beta{}^+ \eta^\beta_{ab} e^a \wedge e^b, \tag{44}$$

because the set of three 2-forms $\pm\eta^\alpha_{ab} e^a \wedge e^b$ is a basis in the space of the (anti)-self-dual forms.

One notes, that (44) can be transformed in (43). Indeed, 1-forms e^a are defined by (42) up to transformation

$$e^a \longrightarrow O^a{}_b e^b, \quad O \in O(4) \tag{45}$$

Let $O = \text{diag}\{1, -1, -1, -1\}$. Then

$$^+\eta^\alpha{}_{ab} O^a_c O^b_d = -\,^-\eta^\alpha{}_{cd} \tag{46}$$

So, redefining e^a and $C^\alpha{}_\beta$ according to (45) and (46), one can transform (44) in (43).

One substitutes (43) in (35). Using the formulas of the section 2.3, one obtains:

$$C^\alpha{}_\gamma C^\beta{}_\gamma = \frac{1}{3}\delta^{\alpha\beta} C^\delta{}_\gamma C^\delta{}_\gamma \tag{47}$$

So

$$C^\alpha{}_\beta = \pm C O^\alpha{}_\beta \tag{48}$$

where

$$O \in SO(3), \quad C = \sqrt{\frac{1}{3} C^\delta{}_\gamma C^\delta{}_\gamma} > 0. \tag{49}$$

For given $O \in SO(3)$ there exists the matrix $\tilde{O} \in SO(4)$ such that

$$^-\eta^\alpha{}_{ab} \tilde{O}^a{}_c \tilde{O}^b{}_d = \left(O^{-1}\right)^\alpha{}_\beta \,^-\eta^\beta{}_{cd} \tag{50}$$

So, redefining e^a according to (45) with $O = \tilde{O}$, and taking into account (48), one reduces (43) to

$$S^\alpha = \pm C \,^-\eta^\alpha{}_{ab} e^a \wedge e^b \tag{51}$$

Finally, substituting (51) in (37), one obtains that $C = 1$. The theorem is proved.

3.2 $SU(3)$ YM action in Plebanski gauge

We start from the usual $SU(3)$ YM action in the first order formalism,

$$S_{YM} = \int \text{tr}\left[F \wedge (dA + A \wedge A) + \lambda^2 F \wedge *F\right], \tag{52}$$

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where F and A are considered as independent variables.

The forms F and A valued in the space of 3×3 anti- Hermitian traceless matrices. So we can write

$$A = \Gamma + i\Phi,$$

$$F = S + iQ, \tag{53}$$

where Γ, Φ, S, and Q valued in the space of real 3×3 matrices, and

$$\Gamma^T = -\Gamma, \quad S^T = -S,$$

$$\Phi^T = \Phi, \quad Q^T = Q,$$

$$\mathrm{tr}\Phi = 0, \quad \mathrm{tr}Q = 0 \tag{54}$$

where the subscript T means transposition.

Substituting (53) in (52), one obtains:

$$S_{YM} = \int \mathrm{tr} \left[S \wedge (R - \Phi \wedge \Phi) + Q \wedge D\Phi + \lambda^2 S \wedge *S - \lambda^2 Q \wedge *Q \right] \tag{55}$$

where

$$R = d\Gamma + \Gamma \wedge \Gamma, \tag{56}$$

$$D\Phi = d\Phi + \Gamma \wedge \Phi + \Phi \wedge \Gamma \tag{57}$$

Decomposition (53) corresponds to certain embedding of the algebra $su(2) \approx o(3)$ in $su(3)$. So Γ and R can be considered as the forms of connection and curvature corresponding to the subgroup $SU(2)$ of the gauge group $SU(3)$ whereas D is covariant derivative defined by the connection Γ.

Due to (54), one can write

$$S^{\alpha\beta} = -\varepsilon^{\alpha\beta\gamma} S^\gamma$$

and to impose the gauge conditions (35) on the 2-forms S^α.

Substituting (39) in (55), one obtains the action

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$$S = 2 \int e^A{}_{C'} \wedge e^{BC'} \wedge R^{AB} + 2 \int e^{AC'} \wedge e^B{}_{C'} \wedge \Phi_A{}^{CDE} \wedge \Phi_{BCDE}$$

$$+ \int Q^{ABCD} \wedge D\Phi^{ABCD} - \lambda^2 \int e^{AC'} \wedge e^B{}_{C'} \wedge *(e_A{}^{D'} \wedge e_{BD'})$$

$$- \lambda^2 \int Q^{ABCD} \wedge Q_{ABCD} \tag{58}$$

where R_{AB} is defined by the formula (25), Φ^{ABCD} and Q^{ABCD} are the forms Φ, Q written in the spinor language[9].

The first term in (58) is the Palatini action. So the action (58) can be considered as one for gravity coupled with several matter fields. In particular, first three terms in the action (58) are invariant under the action of the group of the general coordinate transformations $Diff(R^4)$ - just as in general relativity. But the total action, of course, is not $Diff(R^4)$ invariant because the last two terms in (58) depend on fixed space - time metric via Hodge operator.

We will continue the investigation of $SU(3)$ YM theory in introduced variables in the next section at quantum level. But before we must prove that Plebanski gauge really fixes the gauge up to $SU(2)$ transformations.

3.3   Investigation of the Plebanski gauge.

Any $SU(3)$ matrix U can be written in the form

$$U = e^{i\omega} \tag{59}$$

where ω is a traceless Hermitian 3×3 matrix. Pure imaginary matrices ω corresponds to generators of the subgroup $SU(2)$, whereas real matrices can be considered as coordinates on the space $SU(3)/SU(2)$. Obviously, the latter satisfy the equations

$$\omega^{\alpha\beta} = \omega^{\beta\alpha}, \omega^{\alpha\alpha} = 0. \tag{60}$$

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[9]We have wrote the YM action in new variables for the upper sign case in (39). Below we will show that this is enough to formulate quantum version of the theory under consideration.

Let us consider the infinitesimal gauge transformations with parameters obeying (60):

$$\delta A = id\omega + i[A, \omega] \tag{61}$$

$$\delta F = i[F, \omega] \tag{62}$$

Comparing (61), (62), and (53), one obtains:

$$\delta\Gamma = -[\Phi, \omega], \quad \delta\Phi = D\omega \tag{63}$$

$$\delta S = -[Q, \omega], \quad \delta Q = [S, \omega] \tag{64}$$

Let

$$T^{ABCD} = *\left[S^{(AB} \wedge S^{CD)}\right] \tag{65}$$

Using (64) , one finds

$$\delta T^{ABCD} = c\varepsilon^{mnpq} e^{(A}{}_{A'm} e^{B|A'|}{}_{n} Q^{C}{}_{EFGpq} \omega^{D)EFG} \tag{66}$$

where c is irrelevant numerical constant, $e^{AA'}{}_{m}$ and $Q^{ABCD}{}_{pq}$ are components of the forms $e^{AA'}$ and Q^{ABCD}, whereas ω^{ABCD} is $SU(2)$ spinor that corresponds to $O(3)$ tensor $\omega^{\alpha\beta}$.

To prove that Plebanski gauge reduces the initial $SU(3)$ YM theory to the $SU(2)$ one, it is sufficient to prove that the equations

$$\delta T^{ABCD} = 0 \tag{67}$$

have the only trivial solution $\omega^{ABCD} = 0$ for almost all field configurations.

One notes that due to (60)

$$\omega^{ABCD} = \omega^{(ABCD)} \tag{68}$$

So eqs. (67) are the system of five linear homogenous equations for five unknown ω^{ABCD}.

The system (67) can be rewritten as

$$G^{(ABC}{}_{EFG}\delta^{D)}_{H}\omega^{EFGH} = 0 \tag{69}$$

where

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$$G^{ABCEFG} = \varepsilon^{mnpq} e^A{}_{A'm} e^{BA'}{}_n Q^{CEFG}{}_{pq} \tag{70}$$

In generic case, the spinor G^{ABCEFG} satisfies the only constraints

$$G^{ABCDEF} = G^{(AB)CDEF}$$

$$G^{ABCDEF} = G^{AB(CDEF)} \tag{71}$$

Now let us consider the field configuration for which the only non-zero components of G^{ABCDEF} are G^{000000}, G^{111111}, and

$$G^{000111} = G^{001011} = G^{001101} = G^{001110}.$$

For such configuration one obtains from (69):

$$G^{000000} \omega^{0001} + G^{000111} \omega^{1111} = 0$$

$$G^{000000} \omega^{0000} = 0, \quad G^{000111} \omega^{0011} = 0,$$

$$G^{111111} \omega^{0111} = 0, \quad G^{111111} \omega^{1111} = 0 \tag{72}$$

Obviously, that the system (72) has the only trivial solution for non-zero G^{000000}, G^{111111}, and G^{000111}.

Let

$$M^{ABCD}{}_{EFGH} = G^{(ABC}{}_{(EFG} \delta^{D)}_{H)} \tag{73}$$

the spinor $M^{ABCD}{}_{EFGH}$ can be considered as some 5×5 matrix M. We have proved that $\det M \neq 0$ for certain field configuration. But $\det M$ is polynomial with respect to fields $e^{AA'}$ and Q^{ABCD}. So $\det M \neq 0$ for almost all field configuration. This means that the system (69) has the only trivial solution for almost all configurations of fields.

4 Quantization

We start from usual expression for Euclidean vacuum expectation value of certain Hermitian gauge invariant functional $\mathcal{O} = \mathcal{O}[A, \Psi, \overline{\Psi}]$:

$$< \mathcal{O} >= \int dA d\overline{\Psi} d\Psi \mathcal{O}[A, \overline{\Psi}, \Psi] \exp\{-S_{YM} - S_{mat}\} \tag{74}$$

where $\overline{\Psi}, \Psi$ are matter fields,

$$S_{YM}[A] = -\frac{1}{4\lambda^2} \int \mathrm{tr}(dA + A \wedge A) \wedge *(dA + A \wedge A), \tag{75}$$

and

$$S_{mat} = \int dx \sqrt{g} \left\{ \sum_{flavors} (\overline{\Psi}_f \widehat{\nabla} \Psi_f - m_f \overline{\Psi}_f \Psi_f) \right\} \tag{76}$$

Formula (74) can be written as

$$< \mathcal{O} >= \int dF dA d\overline{\Psi} d\Psi \mathcal{O}[A, \overline{\Psi}, \Psi]$$

$$\exp\{i \int \mathrm{tr}[F \wedge (dA + A \wedge A)]\} \exp\{-\lambda^2 \int \mathrm{tr} F \wedge *F - S_{mat}\} \tag{77}$$

or, finally, as

$$< \mathcal{O} >= \int dS dQ d\Gamma d\Phi d\overline{\Psi} d\Psi \mathcal{O}[\Gamma + i\Phi, \overline{\Psi}, \Psi]$$

$$\exp\{i \int \mathrm{tr}[S \wedge (R - \Phi \wedge \Phi) + Q \wedge D\Phi]\} \exp\{\lambda^2 \int \mathrm{tr}[S \wedge *S - Q \wedge *Q]\} \exp\{-S_{mat}\} \tag{78}$$

where variables S, Q, Γ, and Φ are defined by (53).

We will fix the gauge (namely, Plebanski gauge) by usual Faddeev - Popov trick. We insert in (78) the unit

$$1 = \int_{SU(3)/SU(2)} d\mu(\omega)\delta\left(*[(S^\omega)^{(AB} \wedge (S^\omega)^{CD)}]\right) \Delta_{FP} \tag{79}$$

where $d\mu(\omega)$ is invariant measure on $SU(3)/SU(2)$,$(S^\omega)^{AB}$ is a gauge transformation of S^{AB}, and Δ_{FP} is Faddeev - Popov functional. Then, after usual manipulations, one obtains:

$$< \mathcal{O} >= \int dS dQ d\Gamma d\Phi d\overline{\Psi} d\Psi \mathcal{O}[\Gamma + i\Phi, \overline{\Psi}, \Psi] \delta \left(*[S^{(AB} \wedge S^{CD)}] \right) \det M$$

$$\exp\{i \int \text{tr}[S \wedge (R - \Phi \wedge \Phi) + Q \wedge D\Phi]\} \exp\{\lambda^2 \int \text{tr}[S \wedge *S - Q \wedge *Q]\} \exp\{-S_{mat}\} \tag{80}$$

Here $\det M$ is Faddeev - Popov determinant, where M is 5×5 matrix

$$M^{ABCD}{}_{EFGH} = * \left[S^{(AB} \wedge Q^C{}_{(EFG} \delta^{D)}_{H)} \right] \tag{81}$$

This matrix coincides, on the surface

$$S^{(AB} \wedge S^{CD)} = 0, \tag{82}$$

with the matrix (73). So $\det M \neq 0$ for almost all field configurations (see the section 3.3).

Let $\mathcal{S}_+(\mathcal{S}_-)$ be the set of all 2-forms S^α for which Urbantke metric (37) is positive (negative) definite. We can write the integral (80) as the sum of the integrals over $\mathcal{S}_+$ and $\mathcal{S}_-$.

Obviously, that $\mathcal{S}_-$ is mapped onto $\mathcal{S}_+$ by the transformation $S \longrightarrow -S$, $Q \longrightarrow -Q$. But the latter is equivalent to the complex conjugation in the integral over $\mathcal{S}_-$. So the integral over $\mathcal{S}_+$ is equal to complex conjugated integral over $\mathcal{S}_-$. Hence,

$$< \mathcal{O} >= \text{Re} \int_{\mathcal{S}_+} dS dQ d\Gamma d\Phi d\overline{\Psi} d\Psi \mathcal{O}[\Gamma + i\Phi, \overline{\Psi}, \Psi] \delta \left(*[S^{(AB} \wedge S^{CD)}] \right) \det M$$

$$\exp\{i \int \text{tr}[S \wedge (R - \Phi \wedge \Phi) + Q \wedge D\Phi]\} \exp\{\lambda^2 \int \text{tr}[S \wedge *S - Q \wedge *Q]\} \exp\{-S_{mat}\} \tag{83}$$

We showed in the section 3.1 that the solution of Plebanski gauge conditions (82) for $S \in \mathcal{S}_+$ is given, in vector language, by the formula

$$S^\alpha = {}^-\eta^\alpha{}_{ab} e^a \wedge e^b \tag{84}$$

So, for $S \in \mathcal{S}_+$,

$$\int \prod_{a,n} de_n^a \prod_{\substack{\alpha \\ m>n}} \delta(S^\alpha{}_{mn} - {}^-\eta^\alpha{}_{mn} e_m^a e_n^b)$$

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$$= f(S) \prod_{\alpha \leq \beta} \delta(*[S^\alpha \wedge S^\beta - \frac{1}{3}\delta^{\alpha\beta}S^\gamma \wedge S^\gamma]) \qquad (85)$$

where the function $f(S)$ to be determined.

It is easy to prove that

$$f = const \qquad (86)$$

Indeed, $f(S)$ is scalar density with respect to general coordinate transformations and $O(3)$ gauge transformations. So

$$f = f(\varepsilon^{mnpq}S^\alpha_{\ mn}S^\alpha_{\ pq}) \qquad (87)$$

But the dimension of f is zero. So the function (87) is a constant.

Inserting (85) and (86) in (83), one obtains:

$$< \mathcal{O} >= \mathrm{Re} \int dedQd\,\Gamma d\Phi d\overline{\Psi}d\Psi \mathcal{O}[\Gamma + i\Phi, \overline{\Psi}, \Psi] \det M \exp\{iS_1 - \lambda^2 S_2 - S_{mat}\} \qquad (88)$$

where

$$S_1 = \int e^A_{\ C'} \wedge e^{BC'} \wedge R_{AB} + 2\int e^{AC'} \wedge e^B_{\ C'} \wedge \Phi_A^{\ CDE} \wedge \Phi_{BCDE} + \int Q^{ABCD} \wedge D\Phi_{ABCD},$$

$$S_2 = \int dx \sqrt{g}\,[g^{mp}g^{nq}G_{mn}G_{pq} - (g^{mn}G_{mn})^2] + \int Q^{ABCD} \wedge *Q_{ABCD} \qquad (89)$$

where

$$G_{mn} = e^a_{\ m}e^a_{\ n} \qquad (90)$$

is YM induced metric, $\det M$ is Faddeev - Popov determinant,

$$M^{ABCD}_{\qquad EFGH} = *[e^{(A|C'|} \wedge e^B_{\ C'} \wedge Q^C_{\ (EFG}\delta^{D)}_{H)}] \qquad (91)$$

and

$$S_{mat} = \sum_{flavors} \int dx\sqrt{g}\{i\overline{\Psi}_{fAB}^{\quad(0)}e_a^{\ n}\gamma^a D_n\Psi_f^{\ AB}$$

$$+i\overline{\Psi}_{fAB}^{\quad(0)}e_a^{\ n}\gamma^a\Phi^{ABCD}_{\quad\ \ n}\Psi_{fCD} - m_f\overline{\Psi}_{fAB}\Psi_f^{\ AB}\}$$

In the latter formula $^{(0)}e_a{}^n$ means the space-time tetrad (that is, $g^{mn} = {}^{(0)}e_a{}^m{}^{(0)}e_a{}^n$).

The integrand in (88) is $O(4)$ gauge invariant. To fix this gauge freedom, it is necessary to impose further gauge conditions. The simplest choice is

$$e_{ma} = e_{am} \tag{92}$$

This gauge entangles space - time and gauge degrees of freedom and so, after imposing of the gauge (92), they must be considered on the equal footing.

It is easy to prove, that Faddeev - Popov determinant, corresponding to gauge (92), is equal to $|e|^{\frac{3}{2}}$. So the formula (88) can be written as

$$< \mathcal{O} >= \mathrm{Re} \int\limits_{e_{am} = e_{ma}} de dQ d\,\Gamma d\Phi d\overline{\Psi} d\Psi \mathcal{O}[\Gamma + i\Phi, \overline{\Psi}, \Psi] |e|^{\frac{3}{2}} \det M$$

$$\exp\{iS_1 - \lambda^2 S_2 - S_{mat}\} \tag{93}$$

(It would be remind, that $e^{AA'}{}_m$ and e_{am} in (93) are connected, according to rules of the section 2.2, by relation $e^{AA'}{}_m = g^{aAA'} e_{am}$).

The formula (88) can be also rewritten in manifestly $O(4)$ invariant variables, such as G_{mn},

$$\Phi^{mnpq}{}_r dx^r \equiv \Phi^{\alpha\beta}{}_r \eta^{\alpha}{}_{ab} e^a{}_m e^b{}_n \eta^{\beta}{}_{cd} e^c{}_p e^d{}_q dx^r$$

etc. But in such variables the corresponding action in (88) contains Einstein - Hilbert term $\sqrt{G}R$ and so is not polynomial. So we prefer to consider formulas (88) and (93) as final results of our investigation.

5   Discussion.

We have shown that gravity - like interactions live inside QCD. This conclusion is supported by the results of Ne'eman and Sijacki [21] concerning existence of gravity - like interactions in infrared sector of QCD, and vice-a-versa.

Author hopes that the results presented in this paper will be starting point of various new approaches to QCD. Here we will list only some themes of the further investigations.

- Rescaling fields Q^{ABCD} and G_{mn} in (88), one can rewrite this formula as

$$< \mathcal{O} >=$$
$$= \text{Re} \int dedQd\,\Gamma d\Phi d\bar{\Psi}d\Psi \mathcal{O}[\Gamma + i\Phi, \overline{\Psi}, \Psi] \det M \exp\{\frac{i}{\lambda}S_1 - S_2 - S_{mat}\} \tag{94}$$

So, at least in the weak coupling limit, it is naturally to investigate the functional integral (94) by stationary phase method.

The stationary points are determined by equations

$$\delta S_1 = 0 \tag{95}$$

But equations (95) are nothing but Euclidean Einstein ones. What is the meaning of known exact solutions of Einstein equations (such as gravitational instantons, wormholes, etc.) in the context of QCD?

- In particular, what is the meaning of the flat space solution

$$G_{mn} = c^2 g_{mn}, \quad c = const \tag{96}$$

of the equation (95)? How to construct the expansion of the integrand in (94) near such solution? Does the existence of the flat solutions (96) leads to appearance the vacuum condensates of the gluon fields?

- The actions S_1, S_2, and S_{mat} in (88) are polynomial. So it is possible to derive the corresponding Schwinger equations. What are the solutions of these equations in the usual approximations? Do the solutions exist that correspond to non-zero vacuum condensate of the field G_{mn}?

- The action S_1 in (88) is invariant with respect to the group $Diff(R^4)$ of the general coordinate transformations. It is easy to derive the corresponding Ward identities. Obviously, that these identities express nothing but the energy - momentum conservation. Nevertheless, it is interesting to investigate consequences of such Ward identities because in proposed variables they have very unusual form and, most likely, can lead to new interesting results.

Now let us discuss the shortcomings of the proposed approach. First, our formulation is essencially chiral because left and right $SU(2)$ subgroups of the total $SU(2) \times SU(2) \approx O(4)$ invariance group of the action (58) play the different roles in our formalism. In itself, it is not a difficulty, but after imposing of the gauge conditions that entangle space - time and internal degrees of freedom (as the gauge (92)), one obtains the theory that is not manifestly parity invariant. It is not convenient.

This left - right asymmetry in our approach is connected with the structure of the group $SU(3)$. Indeed, there is no faithful embedding of the group $O(4)$ in $SU(3)$. So it is needed more large gauge group to originate left - right symmetric general relativity-like formalism. So

- it is interesting to develop general relativity-like formalism for the grand unified theories based on the groups $SU(5), SO(10)$, etc. Except left - right symmetric formulation, one may hope to find natural spontaneous parity breaking mechanism in electroweak sector of the theory in this way.

Further, our theory is essential Euclidean and it is unclear how to develop the general relativity-like formulation of YM theory in which YM induced metric has Lorentzian signature in presented approach. This shortcoming again is connected with the structure of the gauge group $SU(3)$. Indeed, the gauge group $SU(3)$ is compact, and so it is impossible to embed in $SU(3)$ neither the non-compact group $SO(3,1)$ nor any its subgroup in a covariant way.

The existence of the only Euclidean formulation of the theory, per se, is not a difficulty. But the formulation of the theory in the Minkowski space is more visual. In particular, the absence of such formulation hampers the investigation of the confinement in our approach. Meanwhile, the results of the works [9, 16] indicates that, may be, there exists black hole like mechanism of the confinement. But black holes live in the Lorentzian space rather then in the Euclidean one.

- Author hopes to overcome above mentioned difficulties by using of formalism developed in the paper [31] where it was shown that $SU(N)$ YM theory is equivalent to certain $GL(N, C)$ gauge theory in the following sense: classes of the gauge equivalent solutions of the initial $SU(N)$

YM theory are in one-to-one correspondence to classes of the gauge equivalent solutions of the above mentioned $GL(N, C)$ gauge theory. In the QCD case $N = 3$, and so Lorentz group $SL(2, C)$ can be embedded in the QCD gauge group in such $GL(3, C)$ formalism. So it is possible to develop the Lorentzian analog of the Euclidean general relativity like formulation of QCD given in the presented work.

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